UNCLASSIFIED

AD NUMBER

AD407437

NEW LIMITATION CHANGE

TO

Approved for public release, distribution unlimited

FROM

Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; 23 MAY 1963. Other requests shall be referred to Air Force Electronic Systems Division, Hanscom AFB, MA.

AUTHORITY

MIT/LL 2 May 1966

THIS PAGE IS UNCLASSIFIED

AD 407 437

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

22G-9

TIME- AND FUEL-OPTIMAL ATTITUDE CONTROL

Michael Athans

[redacted]

23 May 1963

jmk

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the joint support of the U.S. Army, Navy and Air Force under Air Force Contract AF 19(628)-500.

LEXINGTON MASSACHUSETTS

No OTS

ABSTRACT

The problem of attitude control of a body described by the equation $I\ddot{\theta}(t) = \tau(t)$ is examined, where I is the moment of inertia, $\theta(t)$ is the angular displacement, and $\tau(t)$ is the torque exerted by the reaction jets. There exists a magnitude restriction on the torque, i.e., $|\tau(t)| \leq M$ and, in addition, $\theta(t)$ lies in the interval $-\pi \leq \theta(t) \leq \pi$. Under these restrictions the time-optimal and the fuel-optimal control laws are derived and the design of the corresponding optimal controllers is presented.

1. INTRODUCTION

Second order systems described by a differential equation of the form $\ddot{y}(t) = f(t)$, $|f(t)| \leq 1$, have received considerable attention by various investigators. The time-optimal control of such systems is treated in Refs. 1, 2, 3, and 4. Recently, the fuel-optimal control of such systems has been considered. Reference 5 contains the elements of fuel-optimal control with unrestricted response time. References 6 and 7 deal with the fuel-optimal control where the response time is fixed a priori. Reference 8 contains the techniques of fuel-optimal control with an upper bound on the response time.

A common characteristic of the results available to date is the assumption that the (output) variable $y(t)$ can be measured and that its values lie in the interval $-\infty < y(t) < +\infty$. If such is the case, then the problem of forcing $y(t)$ and $\dot{y}(t)$ to zero in minimum time is solved and the solution is well known: one divides the phase plane into two regions by means of the "switch" curve such that in one side $f(t) = +1$ and on the other $f(t) = -1$. Similarly, for the fuel-optimal problem the phase plane is divided into three regions such that in the first region $f(t) = +1$, in the second $f(t) = 0$, and in the third $f(t) = -1$. Moreover, one can design simple nonlinear feedback controllers for minimum-time and minimum-fuel control. It must be emphasized that these results are valid if and only if $-\infty < y(t) < \infty$.

In this paper the attitude control problem of a system described by the differential equation $I\ddot{\theta}(t) = \tau(t)$, $|\tau(t)| \leq M$, $-\pi \leq \theta(t) \leq \pi$ is considered. Note that an additional restriction on the values of the angular displacement variable $\theta(t)$ is imposed. It will be shown that this restriction on $\theta(t)$, which is the result of the inability of the measuring instruments to distinguish between an angle $\theta(t)$ and an angle $\theta(t) \pm 2n\pi$, changes significantly both the time-optimal and the fuel-optimal control laws. Engineering realizations of the optimal control laws will also be given and their operation will be explained.

2. THE PHYSICAL SYSTEM

Consider a body rotating in a plane as illustrated in Fig. 1. The rotation of the body can be controlled by the reaction jets which provide a thrust $f(t)$. Let I denote the moment of inertia of the body about an axis perpendicular to the plane of rotation and passing through the center of mass. Let $\tau(t)$ be the torque produced by the reaction jets, i.e.

$$\tau(t) = c\, f(t) \quad . \tag{1}$$

In the absence of friction, the differential equation of the rotating body is

$$I\ddot{\theta}(t) = \tau(t) \quad . \tag{2}$$

Since the measuring instruments cannot distinguish between an angle $\theta(t)$ and an angle $\theta(t) \pm 2n\pi$, $n = 1, 2, \ldots,$ the angular position $\theta(t)$ is restricted by

$$-\pi \leq \theta(t) \leq \pi \quad . \tag{3}$$

The reaction jets deliver a thrust $f(t)$ which is bounded, i.e.

$$|f(t)| \leq F \quad . \tag{4}$$

Define $u(t)$ by

$$u(t) = \frac{c\, f(t)}{I} \quad . \tag{5}$$

Clearly, $|u(t)| \leq M$, where $M = c\,F/I$. For the sake of simplicity let $M = 1$ and so the magnitude restrictions on the control variable $u(t)$ are

$$|u(t)| \leq 1 \tag{6}$$

for all t. Define the state variables $x_1(t)$ and $x_2(t)$ by

$$x_1(t) = \theta(t) \quad ; \quad x_2(t) = \dot{\theta}(t) \quad . \tag{7}$$

Then the $x_1(t)$ and $x_2(t)$ state variables satisfy the differential equations

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) \end{aligned}\right\} \tag{8}$$

where $x_1(t)$ is restricted by

$$-\pi \leq x_1(t) \leq +\pi \tag{9}$$

for all t.

3. THE TIME-OPTIMAL PROBLEM

The minimum-time control problem for the rotating body is:
Given the system described by Eq. (8) with the control $u(t)$ restricted by Eq. (6) and with the $x_1(t)$ variable restricted by Eq. (9), determine the control which will force any state to the zero state in minimum time.

It is well known that for a second order system, such as the rotating body considered here, the time-optimal control $u(t)$ is either +1 or -1 and it may switch from +1 to -1 or from -1 to +1 at most once (see for example Ref. 4). It is desired to find the value of the control for every possible state (x_1, x_2) where $-\pi \leq x_1 \leq +\pi$.

The method of attack is as follows: A family of curves will be defined which will divide the state plane (i.e. the strip of the $x_1 - x_2$ plane defined by $-\pi \leq x_1 \leq +\pi$, $-\infty < x_2 < \infty$) into several regions. For each point in the state plane the value of the control which will provide the solution to the time-optimal problem will be stated. The elements of the proof of the time-optimality will follow.

Definition 1. Consider the family of curves in the state plane described by the equation

$$x_1 = n\pi - \frac{1}{2} x_2 |x_2| \tag{10}$$

where

$$n = 0, \pm 1, \pm 2, \ldots \tag{11}$$

and

$$-\pi \leq x_1 \leq +\pi \quad . \tag{12}$$

A state (x_1, x_2) which, for a particular value of n, satisfies Eq. (10) is said to belong to the γ_n curve.

Several of the γ_n curves are shown in Fig. 2. Some properties of the γ_n curves will now be developed.

Let $u(t) = \Delta = \pm 1$ in Eq. (8) and solve for $x_1(t)$ and $x_2(t)$ using the initial conditions

$$x_1(0) = \xi_1 \quad ; \quad x_2(0) = \xi_2 \quad . \tag{13}$$

The state variables as functions of time are found to be

$$\left.\begin{aligned} x_1(t) &= \xi_1 + \xi_2 t + \frac{1}{2}\Delta t^2 \\ x_2(t) &= \xi_2 + \Delta t \end{aligned}\right\} \quad . \tag{14}$$

Elimina.ing the time t in the above equations one finds the equation of a trajectory in the state plane

$$x_1 = \xi_1 + \frac{1}{2}\Delta x_2^2 - \frac{1}{2}\Delta \xi_2^2 \quad ; \quad \Delta = \pm 1 \tag{15}$$

which joins the state (ξ_1, ξ_2) with the state (x_1, x_2) when the control is $u(t) = \Delta = \pm 1$.

Consider the portion of the γ_o curve above the x_1 axis, which is labeled γ_{o-} in Fig. 2. The equation of the γ_{o-} curve is obtained from Eq. (9) using $n = 0$ and $x_2 > 0$; therefore

$$x_1 = -\frac{1}{2}x_2^2 \quad . \tag{16}$$

Now in Eq. (15) let $\Delta = -1$ and $x_1 = x_2 = 0$ to find

$$\xi_1 = -\frac{1}{2}\xi_2^2 \quad . \tag{17}$$

Equations (16) and (17) are identical and one concludes that every point on the γ_{o-} curve can be forced to (0, 0) using $u(t) = -1$. Similarly, one finds that every point on the γ_{o+} curve (which is the portion of the γ_o curve below the x_1 axis) can be forced to (0, 0) using $u(t) = +1$. As a matter of fact the γ_o curve is the portion of the switch curve (see Refs. 1, 2, 3, 4) in the strip defined by $-\pi \le x_1 \le +\pi$.

Now consider the γ_2 curve whose equation is

$$x_1 = 2\pi - \frac{1}{2}x_2^2 \tag{18}$$

obtained from Eq. (9) using $n = 2$ and $n_2 > 0$. The γ_2 curve hits the line $x_1 = \pi$ at the point $(\pi, \sqrt{2\pi})$. Now consider the equation of the trajectory obtained from Eq. (15) with $\Delta = -1$, $x_1 = \pi$ and $x_2 = \sqrt{2\pi}$. This trajectory has the equation

$$\xi_1 = 2\pi - \frac{1}{2}\xi_2^2 \quad . \tag{19}$$

Equations (18) and (19) are identical and one concludes that every point on the γ_2 curve can be forced to the point $(\pi, \sqrt{2\pi})$ using $u(t) \pm -1$. However, the points $(-\pi, \sqrt{2\pi})$ and $(\pi, \sqrt{2\pi})$ are equivalent and one may conclude that every point on the γ_2 curve can be forced to (0, 0) using $u(t) = -1$ and that the path is the γ_2 curve until the point

$(\pi, \sqrt{2\pi})$ is reached and then the γ_{0-} curve. Using similar reasoning one establishes the following prepositions.

Preposition 1. Any point on the $\gamma_{0-}, \gamma_2, \gamma_4, \gamma_6, \ldots,$ curves can be forced to $(0, 0)$ using $u(t) = -1$ and the path followed consists of the $\gamma_{0-}, \gamma_2, \gamma_4, \gamma_6, \ldots,$ curves.

Preposition 2. Any point on the $\gamma_{0+}, \gamma_{-2}, \gamma_{-4}, \gamma_{-6}, \ldots,$ curves can be forced to $(0, 0)$ using $u(t) = +1$ and the path followed consists of the $\gamma_{0+}, \gamma_{-2}, \gamma_{-4}, \gamma_{-6}, \ldots,$ curves.

These two prepositions establish properties of the γ_n curves for even values of n.

Definition 2. The statement "apply first the control u^1 and then the control $u^2, \ldots$" will be abbreviated by "apply the control sequence $\{u^1, u^2, \ldots\}$."

The properties of the γ_n curves, for n odd, will now be derived. Consider a state $X = (x_1, x_2)$ in the region between the γ_0 curve and the γ_2 curve as illustrated in Fig. 3. There are two ways of reaching the origin (0, 0) using a control which switches only once. One way is to apply the control sequence $\{-1, +1\}$, i.e. to apply $u(t) = -1$ until the trajectory reaches the γ_{0+} curve at the point Y and then apply $u(t) = +1$ to reach the origin. Let the time required by the control sequence $\{-1, +1\}$ be T_1. The trajectory resulting from the sequence $\{-1, +1\}$ is XYO in Fig. 3. The other way is to apply the control sequence $\{+1, -1\}$, i.e. apply $u(t) = +1$ until the trajectory hits the γ_2 curve at the point Z and then apply $u(t) = -1$ to reach $(0, 0)$ along the γ_2 and γ_{0-} curves (cf Preposition 1). The trajectory resulting from the control sequence $\{+1, -1\}$ is XZAA'O in Fig. 3 and denote the time required by T_2.

It is easy to evaluate the times T_1 and T_2 required by the two methods of control, using Eq. (14). One finds

$$T_1 = x_2 - 2y_2 \tag{20}$$

and

$$T_2 = 2z_2 - x_2 \quad . \tag{21}$$

But from Eqs. (16) and (15)

$$y_2 = -\sqrt{x_1 + \tfrac{1}{2} x_2^2} \tag{22}$$

and from Eqs. (18) and (15)

$$z_2 = \sqrt{-x_1 + \tfrac{1}{2} x_2^2 + 2\pi} \quad . \tag{23}$$

Substituting Eq. (22) into Eq. (20) one finds

$$T_1 = x_2 - 2\sqrt{x_1 + \tfrac{1}{2} x_2^2} \tag{24}$$

and from Eqs. (23) and (21)

$$T_2 = 2\sqrt{-x_1 + \tfrac{1}{2} x_2^2 + 2\pi} \; - x_2 \quad . \tag{25}$$

Now consider the set of states (x_1, x_2) with the property that

$$T_1 = T_2 \quad . \tag{26}$$

Such states must satisfy the equation

$$x_2 - 2\sqrt{x_1 + \tfrac{1}{2} x_2^2} = 2\sqrt{-x_1 + \tfrac{1}{2} x_2^2 + 2\pi} \; - x_2 \tag{27}$$

which simplifies to, after a series of algebraic manipulations,

$$x_1 = \pi - \frac{1}{2} x_2^2 \quad . \tag{28}$$

From Eq. (10), setting $n = 1$ and $x_2 > 0$, the equation of the γ_1 curve is

$$x_1 = \pi - \frac{1}{2} x_2^2 \quad . \tag{29}$$

Equations (28) and (29) are identical and, so, one immediately concludes that the γ_1 curve is a locus of points in the state plane which require the same time to reach the origin using either the control sequence $\{+1, -1\}$, or the control sequence $\{-1, +1\}$.

Using identical reasoning one can compute the locus of all points in the state plane which require the same time to reach the origin using either the control sequence $\{+1, -1\}$ or $\{-1, +1\}$. For example, if one chooses an initial state in the region between the γ_2 and γ_4 curves, one will find that the γ_3 curve is the locus of points which require the same time to reach the origin using either the control sequence $\{+1, -1\}$ or

$\{-1,+1\}$. Thus, one can establish the following preposition.

Preposition 3. Any point on the γ_n curves for odd values of n can be forced to (0, 0) using either the control sequence $\{+1, -1\}$ or the control sequence $\{-1,+1\}$ and requiring the same response time in either case.

Definition 3. Let $R_{m,m+1}$ denote the region which lies between the γ_m and γ_{m+1} curve. More precisely, the state (x_1, x_2) belong to the region (or set) $R_{m,m+1}$ if

$$\left.\begin{array}{l} -\pi \le x_1 \le +\pi \\ \text{and if} \\ 2(m\pi - x_1) < x_2|x_2| < 2((m+1)\pi - x_1) \end{array}\right\} \quad (30)$$

for $m = 0, \pm 1, \pm 2, \ldots,$. For example, the region $R_{-2,-1}$ is the region between the γ_{-2} and the γ_{-1} curve.

The regions $R_{m,m+1}$ are illustrated in Fig. 2.

Theorem 1. Given a state (x_1, x_2) in the state plane. The time-optimal control, that is the control which will force any state (x_1, x_2) to (0, 0) in the shortest possible time is

If $(x_1, x_2) \in R_{m,m+1}$, then $u = (-1)^{m+1}$.
If $(x_1, x_2) \in \gamma_m$ and if $m = 2, 4, 6, \ldots,$ then $u = -1$.
If $(x_1, x_2) \in \gamma_m$ and if $m = -2, -4, -6, \ldots,$ then $u = +1$.
If $(x_1, x_2) \in \gamma_{o-}$, then $u = -1$.
If $(x_1, x_2) \in \gamma_{o+}$, then $u = +1$.
If $(x_1, x_2) \in \gamma_m$ for m odd, then $u = -1$ or $u = +1$ (the time response is the same).

The above theorem provides the value of the time-optimal control for each state in the state plane. The proof of Theorem 1 is given in Appendix 1. The system shown in block diagram form in Fig. 4 represents an engineering realization of the control law given by the theorem and it will result in the minimum response time control of the rotating body.

The design and operation of the system of Fig. 4 is based on the following

reasoning. Define a function h_n by

$$h_n = x_1 - n\pi + \frac{1}{2} x_2 |x_2| \quad . \tag{31}$$

Note that if

$$h_n = 0 \tag{32}$$

then the state (x_1, x_2) belongs to the γ_n curve (see Definition 1). If

$$h_n > 0 \tag{33}$$

then the state (x_1, x_2) is <u>above</u> the γ_n curve and, finally, if

$$h_n < 0 \tag{34}$$

then the state (x_1, x_2) is <u>below</u> the γ_n curve. As indicated in Fig. 4, the signals h_n are generated from the x_1 and x_2 state variables and a constant source providing the signal π. The only nonlinearity required is of the square law type, indicated by the block N in Fig. 4. The nonlinearity N is such that its output signal is $\frac{1}{2} x_2 |x_2|$ when the input signal is x_2. From the signal h_o the system generates all other signals h_n by repeated addition or subtraction of π.

Suppose that the measured state (x_1, x_2) is in the region $R_{m, m+1}$, that is the state lies between the γ_m and γ_{m+1} curves. Then the signals h_n will be:

$$\left.\begin{array}{lll} h_i > 0 & \text{for all} & i \leq m \\ h_i < 0 & \text{for all} & i \geq m+1 \end{array}\right\} \quad . \tag{35}$$

For clarity, suppose that the signals $h_2, h_3, h_4, \ldots,$ are negative and that the signals $h_1, h_0, h_{-1}, h_{-2}, \ldots,$ are positive, as illustrated in Fig. 4. Then the measured state point (x_1, x_2) is between the γ_1 and γ_2 curve, i.e. it belongs to the Region $R_{1,2}$. As shown in Fig. 4 each h_i signal activates a polarized relay with outputs +1 and -1. Let q_i be the output of the i^{th} relay, i.e. $q_i = \text{sgn}\{h_i\} = \pm 1$. For example, suppose that in Fig. 4 $q_2 = q_3 = q_4 = \ldots = -1$ and $q_1 = q_0 = q_{-1} = q_{-2} = \ldots = +1$. Connect the coils of <u>normally open</u> relays r_i between adjacent signals q_i and q_{i+1}, where the contact of the r_i relay is connected to a voltage $(-1)^{i+1}$ and the outputs of all the relays

are connected to a common terminal. Only one of the relays r_i will be activated. For example, in the specific situation discussed above only the relay r_1 will be closed and all the other relays will remain open, because there exists a voltage difference only across the coil of the relay r_1 and there is no voltage drop across the coils of all the other relays. The signal at the common terminal will have the correct polarity to activate the relay-like mechanism of the gas jets which will now produce the correct thrust to drive θ and $\dot{\theta}$ to zero in minimum time.

In any physical situation one can expect a bound on the magnitude of the angular velocity x_2; therefore the nonlinear computer of Fig. 4 would be terminated at the required number of stages.

4. THE FUEL-OPTIMAL PROBLEM

The minimum-fuel attitude control problem for the rotating body is to drive $\theta(t)$ and $\dot{\theta}(t)$ to zero using as little gas (or fuel) as possible. More precisely:

Given the system described by Eq. (8) with the control $u(t)$ restricted in magnitude by Eq. (6) and with the $x_1(t)$ state variable restricted by Eq. (9), determine the control which will force any state to the zero state with minimum consumed fuel F measured by

$$F = \int_0^T k \, |u(t)| \, dt \tag{36}$$

where T is the response time (unspecified) and k is a positive constant of proportionality. Furthermore, if more than one control requires the same minimum fuel, then find the fuel-optimal control which requires the least response time.

It is well known (see Refs. 5, 6, 7, 8) that for a second order system, such as the rotating body considered in this paper, the fuel-optimal control $u(t)$ is either +1, 0, or -1 and that the only control sequences (see Definition 2) that may be used are $\{0\}$, $\{+1\}$, $\{-1\}$, $\{+1, 0\}$, $\{-1, 0\}$, $\{0, +1\}$, $\{0, -1\}$, $\{+1, 0, -1\}$ and $\{-1, 0, +1\}$.

Consider the family of the γ_n curves (see Definition 1) described by

$$x_1 = n\pi - \frac{1}{2} x_2 |x_2| \tag{37}$$

where

$$n = 0, \pm 2, \pm 4, \pm 6, \dots, \tag{38}$$

and

$$-\pi \leq x_1 \leq +\pi \quad . \tag{39}$$

This family of γ_n curves (n even) is shown in Fig. 5.

The fuel-optimal control law is provided by the theorem below.

Theorem 2. Given a state (x_1, x_2) in the state plane.

If (x_1, x_2) belongs to the γ_n curve for $n = 2, 4, 6, \dots$, then $u = -1$.

If (x_1, x_2) belongs to the γ_n curve for $n = -2, -4, -6, \ldots,$ then $u = +1$.

If (x_1, x_2) belongs to the γ_{o+} curve, then $u = +1$.

If (x_1, x_2) belongs to the γ_{o-} curve, then $u = -1$.

For all other points in the state plane $u = 0$.

The proof of this theorem is very similar to the proof presented in Ref. 5 and is, therefore, omitted for the sake of brevity. Figure 6 illustrates an engineering realization of the control law provided by Theorem 2. The signals h_n, for $n = 0, \pm 2, \pm 4, \ldots,$ are generated as explained before. If the state point belongs to the γ_i curve, then the h_i signal is zero. In Fig. 6 the signals h_o and x_1 are used to generate the two signals h_{o-} and h_{o+}; if $h_{o-} = 0$, then (x_1, x_2) belongs to the γ_{o-} curve and if $h_{o+} = 0$, then (x_1, x_2) belongs to the γ_{o+} curve. Each signal h_i, $i = \pm 2, \pm 4, \ldots,$ and h_{o-} and h_{o+} activate normally closed relays so that a voltage of -2 volts appears on the common terminal if the state point is on the $\gamma_{o-}, \gamma_2, \gamma_4, \ldots,$ curves and a voltage of +2 volts appears only if the state belongs to the $\gamma_{o+}, \gamma_{-2}, \gamma_{-4}, \ldots,$ curves. For any other state the relays are open and no voltage appears at the terminal. Thus, the signals at the terminal can be used to open or close the reaction jets on the body and to deliver the correct thrust for minimum fuel operation.

If the bound on the magnitude of the angular velocity x_2 is known, the computer of Fig. 6 would again have a finite number of stages.

5. DISCUSSION OF THE RESULTS

In the two previous sections the time-optimal and the fuel-optimal control laws were derived and the corresponding controllers were designed. In this section, equations for the response time and for the consumed fuel required by the time-optimal and by the fuel-optimal systems are presented.

Let t^* be the minimum time to force a given state (x_1, x_2) to $(0,0)$. The minimum time t^* is determined from the formulae below:

$$t^* = \begin{cases} x_2 - 2\sqrt{x_1 + \frac{1}{2}x_2^2 - m\pi} & \text{if } (x_1, x_2) \in R_{m,m+1} \text{ and } m = 0, 2, 4, 6, \ldots \\ -x_2 + 2\sqrt{-x_1 + \frac{1}{2}x_2^2 + m\pi} & \text{if } (x_1, x_2) \in R_{m,m+1} \text{ and } m = 1, 3, 5, 7, \ldots \\ x_2 + 2\sqrt{x_1 + \frac{1}{2}x_2^2 - m\pi} & \text{if } (x_1, x_2) \in R_{m,m+1} \text{ and } m = -2, -4, -6, \ldots \\ -x_2 - 2\sqrt{-x_1 + \frac{1}{2}x_2^2 + m\pi} & \text{if } (x_1, x_2) \in R_{m,m+1} \text{ and } m = -1, -3, -5, \ldots \end{cases} \tag{40}$$

The fuel F^* consumed by the time-optimal system is given by

$$F^* = kt^* \tag{41}$$

since the fuel is measured by Eq. (36) and for the time-optimal system $|u(t)| = 1$ and the response time is t^*.

Let T be the response time required by the fuel-optimal system. The response time T, as a function of x_1 and x_2, is given below:

If

$$x_2 > 0 \tag{42}$$

find the value of m, $m = 0, 2, 4, \ldots,$ such that

$$\sqrt{2\pi(m-1)} \le x_2 \le \sqrt{2\pi(m+1)} \quad . \tag{43}$$

Using the value of the determined m the response time is

$$T = \begin{cases} \dfrac{1}{x_2}\{\tfrac{1}{2}x_2^2 + m\pi - x_1\} & \text{if} \quad x_1 \le m\pi - \tfrac{1}{2}x_2^2 \\[2ex] \dfrac{1}{x_2}\{\tfrac{1}{2}x_2^2 + (m+2)\pi - x_1\} & \text{if} \quad x_1 > m\pi - \tfrac{1}{2}x_2^2 \end{cases} \quad . \tag{44}$$

If

$$x_2 < 0 \tag{45}$$

find the value of m, $m = 0, -2, -4, \ldots,$ such that

$$-\sqrt{-2\pi(m-1)} \le x_2 \le -\sqrt{-2\pi(m+1)} \quad .$$

Using the value of m the response time is

$$T = \begin{cases} \dfrac{1}{x_2}\{-\tfrac{1}{2}x_2^2 + m\pi - x_1\} & \text{if} \quad x_1 \ge m\pi + \tfrac{1}{2}x_2^2 \\[2ex] \dfrac{1}{x_2}\{-\tfrac{1}{2}x_2^2 + (m-2)\pi - x_1\} & \text{if} \quad x_1 < m\pi + \tfrac{1}{2}x_2^2 \end{cases} \quad . \tag{46}$$

Let F_o denote the fuel consumed by the fuel-optimal system. The fuel F_o is given by

$$F_o = k|x_2| \tag{47}$$

and, of course, F_o represents the minimum fuel required to force (x_1, x_2) to $(0, 0)$. From Eqs. (42) and (45) one finds

$$\lim_{|x_2| \to 0} T = \infty \quad , \tag{48}$$

$$\lim_{|x_2| \to 0} F_o = 0 \quad . \tag{49}$$

The large response times required by the fuel-optimal system for small values of x_2 are undesirable from an engineering point of view. There exist techniques for improving the time response of such a system with an increase in the amount of fuel required. Such techniques have been reported in Ref. 8 and they can be used to improve the

response of the fuel-optimal system.

Another disadvantage of the fuel-optimal system and its engineering realization of Fig. 6 is that switching from $u = 0$ to $u = +1$ or -1 occurs only at the γ_n curves. This means that a slight malfunction of the relays in Fig. 6 might cause the control to remain at $u = 0$ and thus impair the time-response of the system, but not affect the consumed fuel. An engineering cure for the phenomenon is to use relays with some dead-zone or hysteresis so as to "widen" the γ_n curves in the plane so that the transition from "off" to "on" occurs in finite areas rather than on curves.

6. CONCLUSIONS

The time-optimal and the fuel-optimal control laws have been derived for the attitude control of a satellite. The necessary nonlinear controllers were designed and their operation explained.

APPENDIX

Proof of Theorem 1

We shall use the Maximum Principle (see Ref. 4) to prove Theorem 1. The time-optimal Hamiltonian for the system of Eq. (8) is

$$H = 1 + x_2(t)p_1(t) + u(t)p_2(t) \quad , \tag{49}$$

The minimum of H occurs at

$$u(t) = - \operatorname{sgn}\{p_2(t)\} \quad . \tag{50}$$

But

$$\dot{p}_1 = - \frac{\partial H}{\partial x_1(t)} = 0 \tag{51}$$

$$\dot{p}_2(t) = - \frac{\partial H}{\partial x_2(t)} = -p_1(t) \quad . \tag{52}$$

Let

$$p_1(0) = \pi_1 \quad ; \quad p_2(0) = \pi_2 \quad . \tag{53}$$

Then

$$p_2(t) = \pi_2 - \pi_1 t \quad . \tag{54}$$

Therefore

$$u(t) = - \operatorname{sgn}\{\pi_2 - \pi_1 t\} \quad . \tag{55}$$

Equation (55) means that only the four control sequences (see Definition 2) can be candidates for time-optimal control

$$\{+1\},\ \{-1\},\ \{+1, -1\},\ \{-1, +1\} \quad . \tag{56}$$

We shall first prove that <u>if $(x_1, x_2) \epsilon \gamma_{0+}$, then $u = +1$ is the time-optimal control,</u> i.e. the sequence $\{+1\}$ is time-optimal. Consider, in Fig. 7, the point $A \epsilon \gamma_{0+}$. By definition, the control sequence $\{+1\}$ transfers A to O along AO (i.e. along the γ_{0+} curve). From the remaining sequences of Eq. (56) only the sequence $\{-1, +1\}$ can force

A to O along the trajectory ABCC'AO, provided the transition from -1 to +1 occurs at $B \epsilon \gamma_{-2}$. But the time along ABCC'AO is greater than the time along AO (because AO is a portion of ABCC'AO), hence the sequence $\{-1, +1\}$ is not time-optimal, and, hence $\{+1\}$ is time-optimal. Q.E.D. Using similar reasoning we can prove that <u>if $(x_1, x_2) \epsilon \gamma_{0^-}$, then $u = -1$ in the time-optimal control.</u>

Now suppose the initial state $(x_1, x_2) \epsilon R_{0,1}$. The only control sequences that can force (x_1, x_2) to $(0, 0)$ are $\{-1, +1\}$ and $\{+1, -1\}$. The sequence $\{-1, +1\}$ is the time-optimal one; this follows from the arguments presented in Section 3 where the γ_1 curve was defined (that is, the arguments following Definition 2 up to Preposition 3).

For all the other states the procedure is quite similar. First one finds the control sequences which force (x_1, x_2) to $(0, 0)$ and then by direct computation one can prove all the statements of Theorem 1.

ACKNOWLEDGMENT

The author is indebted to Dr. Harold K. Knudsen of Lincoln Laboratory for the many stimulating discussions.

REFERENCES

1. A.M. Hopkin, "A Phase Plane Approach to the Design of Saturating Servomechanisms," Trans. AIEE, 70 (1950), pp. 631-639.

2. D.C. McDonald, "Nonlinear Techniques for Improving Servo Performance," Proc. National Electronics Conference, 6 (1950), pp. 400-421.

3. I. Bogner, "An Investigation of the Switching Criteria for Higher-Order Contractor Servomechanisms," Cook Research Labs Interim Report, PR-16-9 (1953).

4. L.S. Pontryagin, et al, The Mathematical Theory of Optimal Processes (Interscience Publishers, J. Wiley and Sons, New York 1962).

5. M. Athans, "Minimum-Fuel Feedback Control Systems: Second Order Case," IEEE Trans. on Applications and Industry, 82 (March 1963), pp. 8-17.

6. I. Flügge-Lotz and H. Marbach, "The Optimal Control of Some Attitude Control Systems for Different Performance Criteria," ASME Paper 62-JACC-6 (1962).

7. J.S. Meditch, "On Minimal Fuel Satellite Attitude Control," accepted for presentation at 1963 JACC.

8. M. Athans, "Fuel-Optimal Control of a Double Integral Plant with Response Time Constraints," accepted for publication in Trans. IEEE (1963).

FIGURE CAPTIONS

Fig. 1	The physical system.
Fig. 2	The γ_n curves and the $R_{n,n+1}$ regions.
Fig. 3	The trajectories XZAA'O and XYO.
Fig. 4	An engineering realization of the time-optimal control law provided by Theorem 1.
Fig. 5	The γ_n curves ($n = 0, \pm 2, \pm 4, \ldots$). ABO and CDD'EFF'O are examples of fuel-optimal trajectories.
Fig. 6	An engineering realization of the fuel-optimal control law provided by Theorem 2.
Fig. 7	The trajectories which arise from different control sequences.

3-22-4348

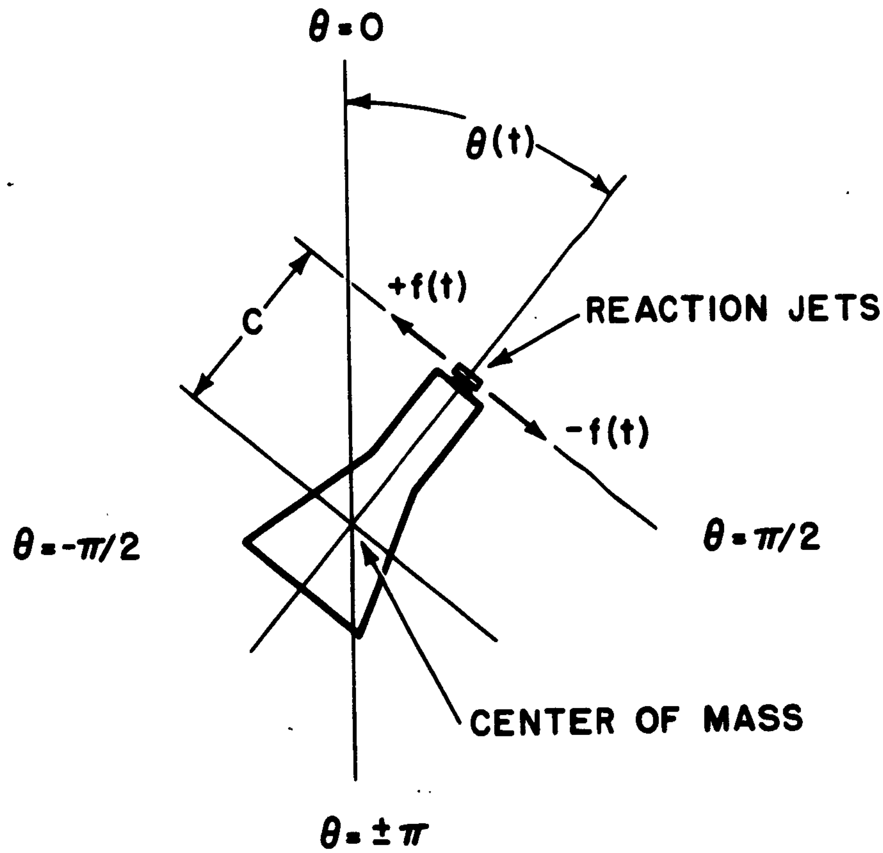


Fig. 1

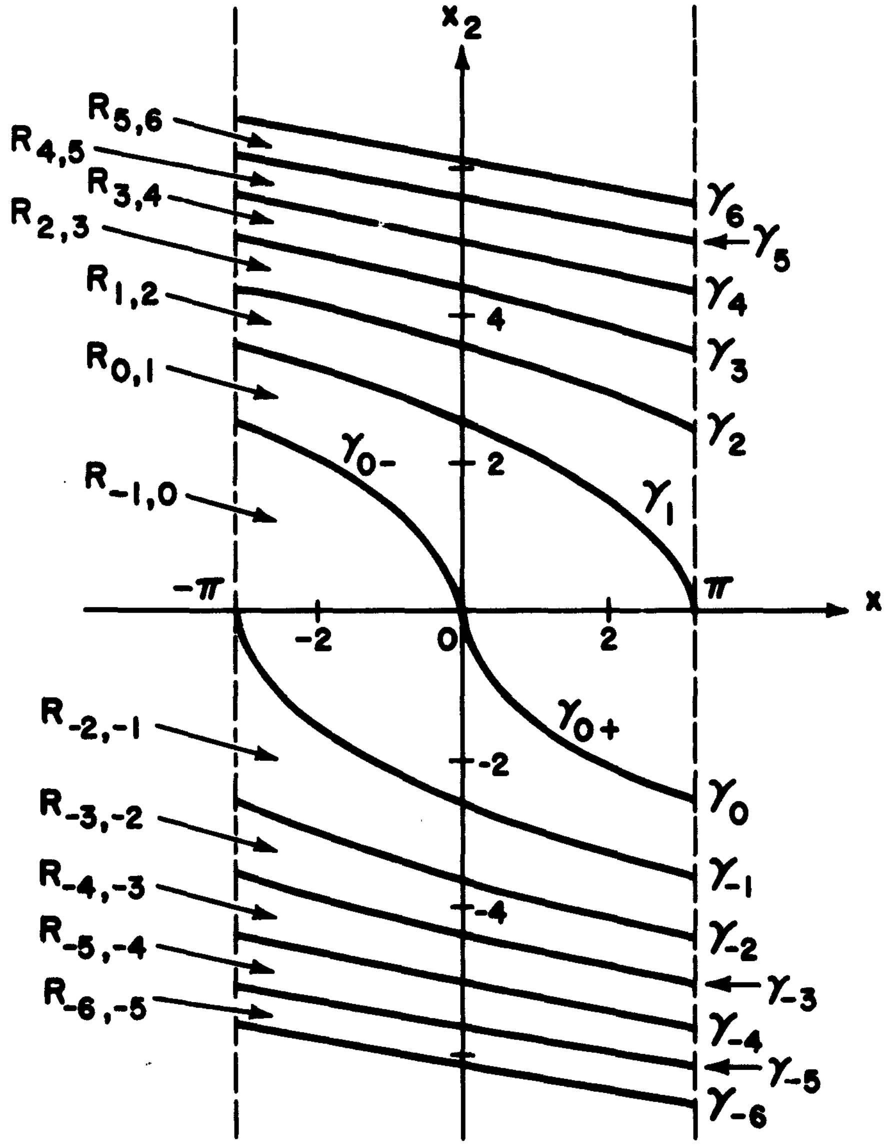

x_2
x_1
$R_{5,6}$
$R_{4,5}$
$R_{3,4}$
$R_{2,3}$
$R_{1,2}$
$R_{0,1}$
$R_{-1,0}$
$R_{-2,-1}$
$R_{-3,-2}$
$R_{-4,-3}$
$R_{-5,-4}$
$R_{-6,-5}$
γ_6
γ_5
γ_4
γ_3
γ_2
γ_1
γ_{0-}
γ_{0+}
γ_0
γ_{-1}
γ_{-2}
γ_{-3}
γ_{-4}
γ_{-5}
γ_{-6}
$-\pi$
π
4
2
-2
0
2
-2
-4

Fig. 2

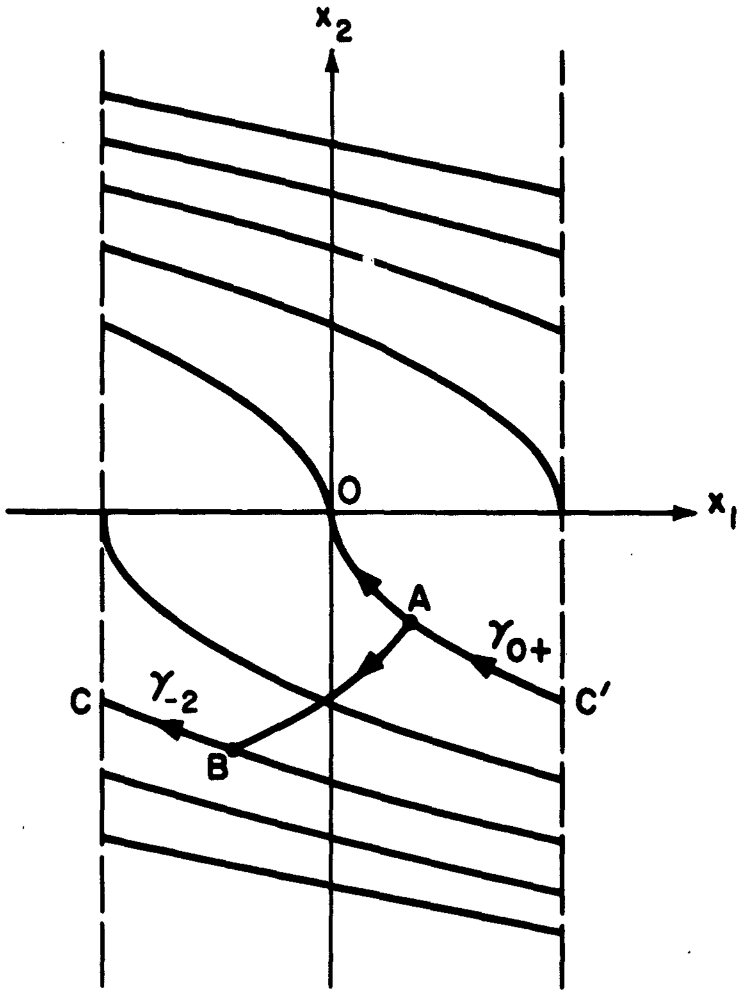
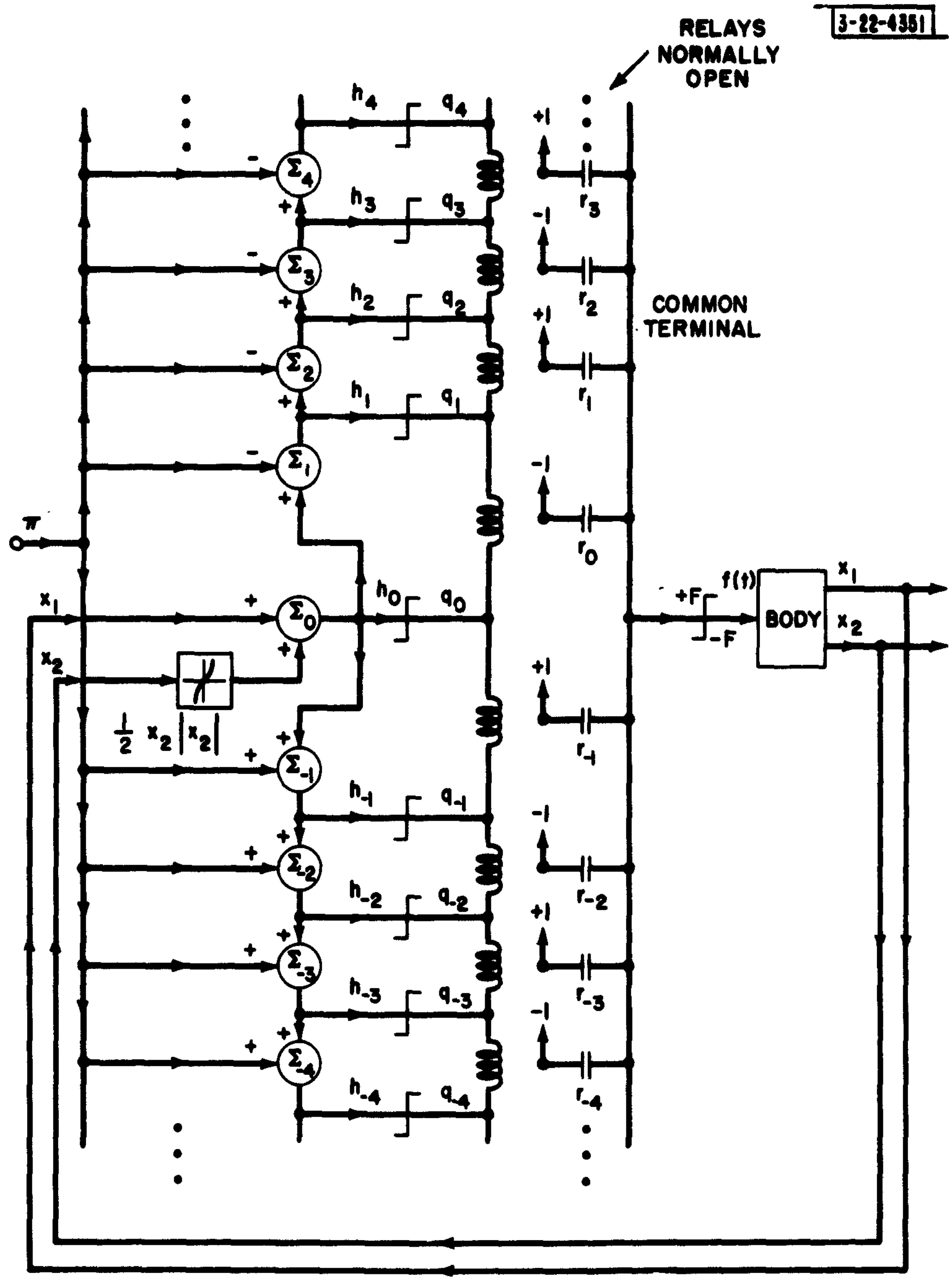


Fig. 3



3-22-4351
RELAYS
NORMALLY
OPEN
COMMON
TERMINAL
BODY
f(t)
+F
-F
π
x1
x2
1/2 x2 |x2|

Fig. 4

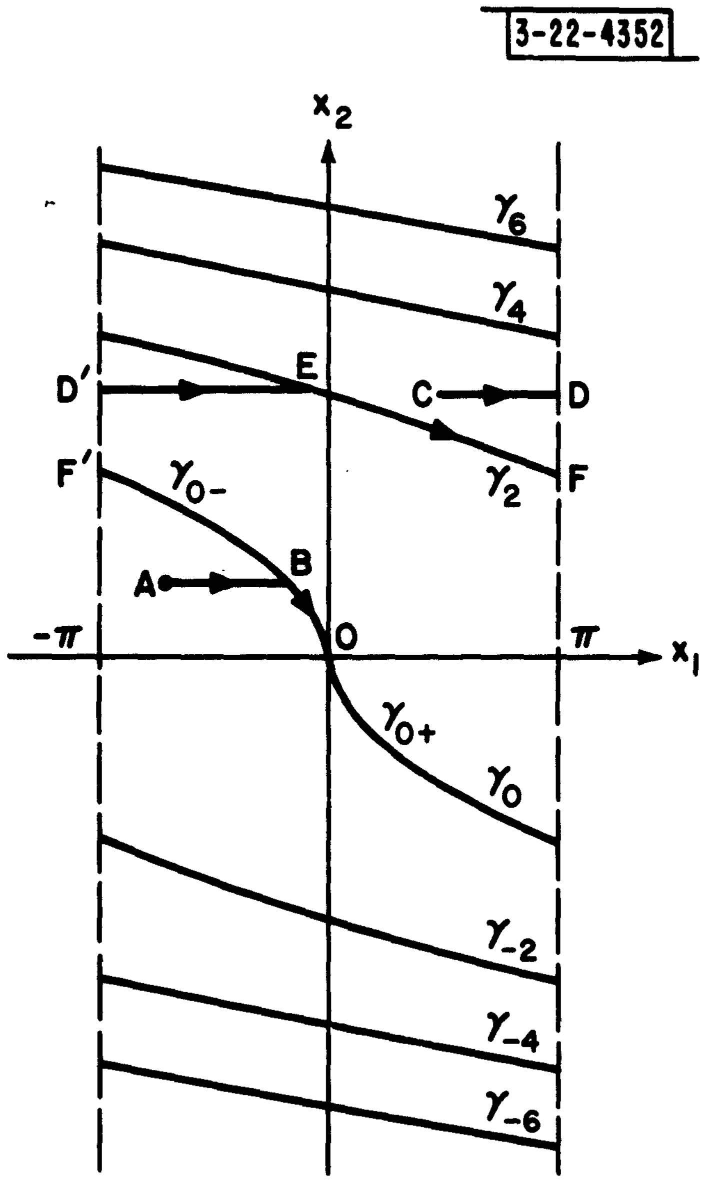
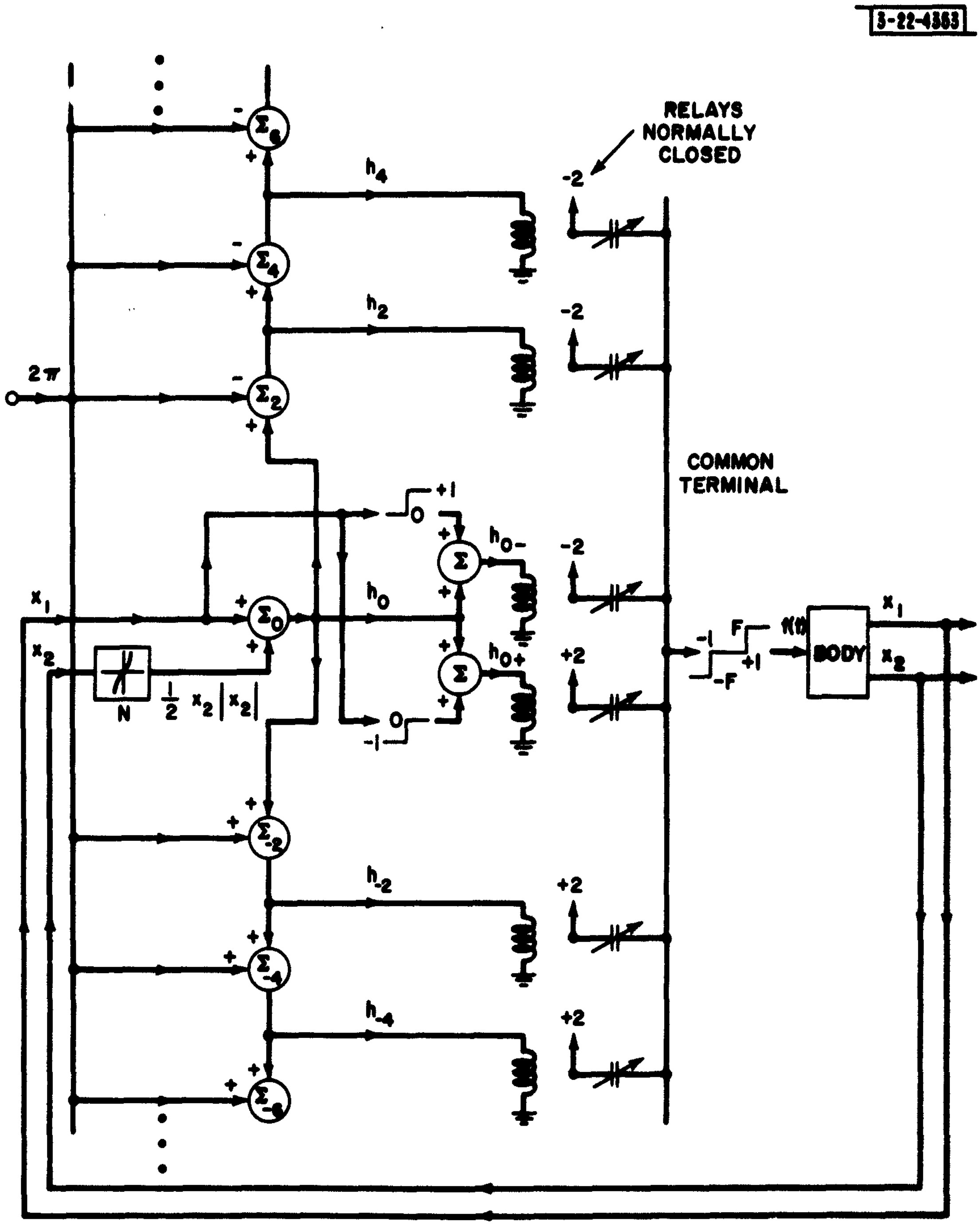

3-22-4352
x_2
γ_6
γ_4
E
D′
C
D
γ_2
F′
F
γ_{0-}
B
A
O
$-\pi$
π
x_1
γ_{0+}
γ_0
γ_{-2}
γ_{-4}
γ_{-6}

Fig. 5



3-22-4363
RELAYS
NORMALLY
CLOSED
COMMON
TERMINAL
Σ_6
Σ_4
Σ_2
Σ_0
Σ_{-2}
Σ_{-4}
Σ_{-6}
h_4
h_2
h_0
h_{0-}
h_{0+}
h_{-2}
h_{-4}
2π
x_1
x_2
N
$\frac{1}{2}\ x_2|x_2|$
-2
-2
-2
+2
+2
+2
+1
-1
F
-F
f(t)
BODY

Fig. 6

3-22-4350

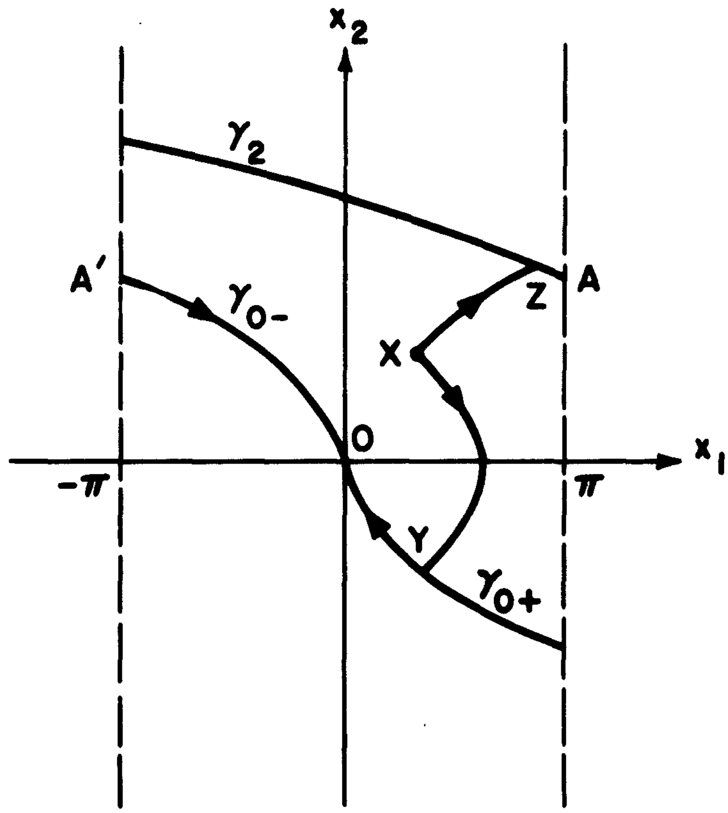


Fig. 7

DISTRIBUTION LIST

Division 2

C.R. Wieser
S.H. Dodd

Group 22

V.A. Nedzel
J.F. Nolan
W.I. Wells

C.R. Arnold
M. Athans (100)
T.C. Bartee
P.L. Falb
A. Freed
L.A. Gardner, Jr.
A.A. Grometstein
J.H. Halberstein
L. Kleinrock
H.K. Knudsen
E. Korngold
H.J. Kushner
J. Lewis
D. Sakrison
M. Schneider
F.C. Schweppe
C.W. Uskavitch
J.M. Winett
P. Wood

Group 25

H. Sherman

Group 28

H.E. Meily
A.J. Morency
J. Slesers

Dr. R. E. Kalman
RIAS Inc.
7212 Bellona Avenue
Baltimore 12, Maryland

Professor C. A. Desoer
Dept. of Electrical Engineering
University of California
Berkeley 4, California

Professor O. J. M. Smith
Dept. of Electrical Engineering
University of California
Berkeley 4, California

Professor E. Polak
Dept. of Electrical Engineering
University of California
Berkeley 4, California

Professor O. Elgerd
Dept. of Electrical Engineering
University of Florida
Gainsville, Florida

Professor Y. C. Ho
Computation Laboratory
Harvard University
Cambridge, Massachusetts

Dr. M. Masarovi
Case Institute of Technology
Cleveland 6, Ohio

Professor S. S. L. Chang
Dept. of Electrical Engineering
New York University
New York, New York

Professor I. Flügge-Lotz
Dept. of Engineering
Stanford University
Stanford, California

Dr. George Lendaris, Defense Systems Div.
General Motors Corp.
Santa Barbara, California

Dr. C. Harvey
Minneapolis Honeywell Reg. Co.
Minneapolis, Minnesota

Mr. Ernest Holzmann
Rm. 116, Bldg. 3, General Electric Co.
Electronics Lab., P.O. Box 1122
Syracuse, New York

Professor R. Bach
112 G. Northeastern University
Huntington Avenue
Boston, Massachusetts

Dr. M. Aoki
Dept. of Engineering
University of California
Los Angeles, California

Dr. E. B. Lee
Minneapolis Honeywell Reg. Co.
Minneapolis, Minnesota

Professor George Newton
Dept. of Electrical Engineering
Massachusetts Institute of Technology
Cambridge, Massachusetts

Professor James Eaton
University of California
Berkeley 4, California

Dr. Lucien Neustadt
Aerospace Corporation
El Segundo, California

Hq. ESD (Maj. S. Jeffery ESRC)
L. G. Hanscom Field
Bedford, Massachusetts

Hq. ESD (Lt. J. Egan)
L. G. Hanscom Field
Bedford, Massachusetts

Also distribute to NASA Distribution List.